

# Binary periodic synchronizing sequences

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## Abstract

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In this article, we consider words over  $\{0, 1\}$ . The *autodistance* of such a word is the lowest among the Hamming distances between the word and its images by circular permutations other than identity; the word's *reverse autodistance* is the highest among these distances. For each  $l \geq 2$ , we study the words of length  $l$  whose autodistance and reverse autodistance are close to  $l/2$  (we call such words *synchronizing sequences*).

We establish, for every  $l \geq 3$ , an upper bound on the autodistance of words of length  $l$ . This upper bound, called  $up(l)$ , is very close to  $l/2$ .

We briefly describe the maximal-period linear recurring sequences, a previously known family of words over  $\{0, 1\}$ ; such words exist for every length of the form  $l = 2^n - 1$  and their autodistances achieve the upper bound  $up(l)$ .

Examples of words whose autodistance and reverse autodistance are both equal or close to  $up(l)$  are discussed; we describe the method (based on simulated annealing) which was used to find the examples.

We prove that, for sufficiently large  $l$ , an arbitrarily high proportion of words of length  $l$  will have both their autodistance and reverse autodistance very close to  $up(l)$ .

## 1. Introduction

### 1.1. Subject of the article

Modern radio techniques, including radar and spread-spectrum communications, make use of finite sequences of bits exhibiting various *correlation properties* (e.g. [5], [2] Chapters 10 and 12, [1]). The correlation properties of a sequence measure how easily it can be distinguished, after a transmission with errors, from other related sequences (the notion of *related sequences* is application-dependent).

We study here two correlation properties, the *autodistance* and the *reverse autodistance*. The autodistance measures how well, in the worst case, the receiver will be able to distinguish between the sequence and a nonidentical circular permutation of it (in this case, we consider that circular permutations of a sequence are related to it). The reverse autodistance measures the difficulty that the receiver will have, in the worst case, distinguishing between the sequence and a circular permutation of its one's complement (here, we consider that circular permutations of the one's complement of a sequence are related to the sequence).

In this study, we focus on searching for, and estimating the number of, sequences that exhibit a high autodistance (the *synchronizing sequences*) and those that exhibit both a high autodistance and a low reverse autodistance (the *double synchronizing sequences*).

## 1.2. Contents

Section 2 of the article introduces the necessary notation and mathematical objects (including precise definitions of autodistance and reverse autodistance).

In Section 3, we investigate which values the autodistance and reverse autodistance can attain. We establish, for each length  $l$ , an upper bound on the autodistance of sequences of this length (Section 3.1); we complete this basic result with several remarks about the reverse autodistance of certain classes of sequences (Sections 3.2–3.3).

In Sections 4–6, we either find, or prove the existence of, sequences whose autodistance and reverse autodistance approach the previously established bounds.

In Section 4, quoting already known results [4], we introduce the *maximal-period linear recurring sequences*, a family of double synchronizing sequences which achieve the bounds for certain lengths  $l$ .

In Section 5, we describe examples of double synchronizing sequences whose lengths are between 3 and 405; these examples achieve, or almost achieve, the bounds. We present a computational method, based on *simulated annealing*, which we used to find the examples.

In Section 6, we establish a theorem implying that among very long sequences of bits, almost all have their autodistances and reverse autodistances close to the respective bounds.

## 2. Definitions and notation

### 2.1. Basic notation

- $i \sqcap j$       greatest common divisor (GCD) of  $i, j \in \mathbb{N}$ .
- $[a..b]$     interval  $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$ .
- $[a..b)$     interval  $\{i \in \mathbb{Z} \mid a \leq i < b\}$ .

- $\mathbf{N}_{2+}$  set of natural numbers  $\geq 2$ .
- $\{0, 1\}^{2+}$  set of words over  $\{0, 1\}$  of length  $\geq 2$ .
- $\{0, 1\}^l$  for  $l \in \mathbf{N}_{2+}$ , set of words over  $\{0, 1\}$  of length  $l$ .
- $|S|$   $|E|$  length of the word  $S \in \{0, 1\}^{2+}$ ; cardinality of the set  $E$ .
- $|S|_0$   $|S|_1$  number of *zeros* (resp. *ones*) in  $S \in \{0, 1\}^{2+}$ .
- $(F_x)_{x \in X}$  the family of elements  $F_x$ , indexed by elements  $x \in X$ ; by definition,  $|(F_x)_{x \in X}| = |X|$ .
- $|\mathcal{F}|_A$  number of elements of the family  $\mathcal{F}$  belonging to the set  $A$ ; if  $\mathcal{F} = (F_x)_{x \in X}$ , then
 
$$|\mathcal{F}|_A = |\{x \in X \mid F_x \in A\}|.$$
- $A \triangle B$  symmetrical difference between sets:  $A \triangle B = (A \cup B) - (A \cap B)$ .
- $xA$  for  $x \in \mathbf{R}$  and  $A \subset \mathbf{R}$ , the set  $\{xy \mid y \in A\}$ ; the definitions of  $A + x$  and  $A - x$  are analogous.
- $S[i]$  for  $S \in \{0, 1\}^{2+}$  and  $0 \leq i < |S|$ , the  $i$ th digit of  $S$ .
- $\tau_p$  circular permutation by  $p$  of words from  $\{0, 1\}^{2+}$ :
 
$$\tau_p(S)[i] = S[(i + p) \bmod |S|].$$
- $d(S, T)$  for  $S, T \in \{0, 1\}^l$ , the Hamming distance between  $S$  and  $T$ :
 
$$d(S, T) = |\{i \in [0..l) \mid S[i] \neq T[i]\}|.$$

## 2.2. Notation of objects defined in the article

- $d(S)$  for  $S \in \{0, 1\}^{2+}$ , the autodistance of  $S$  (Definition 2.1).
- $d'(S)$  for  $S \in \{0, 1\}^{2+}$ , the reverse autodistance of  $S$  (Definition 2.2).
- $\text{up}(l)$  for  $l \in \mathbf{N}$ ,  $l \geq 3$ ,  $\text{up}(l) = 2 \lfloor (l + 1)/4 \rfloor$  (Definition 3.2).

## 2.3. Autodistance and synchronizing sequences

**Definition 2.1** (*autodistance*). For  $S \in \{0, 1\}^{2+}$ , the *autodistance* of  $S$  is the minimum of the Hamming distances between  $S$  and all its images by circular permutations other than identity:

$$d(S) = \min_{p \in [1..|S|)} d(S, \tau_p(S)).$$

**Definition 2.2** (*reverse autodistance*). For  $S \in \{0, 1\}^{2+}$ , the *reverse autodistance* of  $S$  is the maximum of the Hamming distances between  $S$  and all its images by circular permutations:

$$d'(S) = \max_{p \in [0..|S|)} d(S, \tau_p(S)).$$

*Examples:* The null word of any length satisfies  $d(S) = d'(S) = 0$ . The words 001 and 0011 satisfy

$$d(001) = d'(001) = 2,$$

$$d(0011) = 2,$$

$$d'(0011) = 4.$$

**Definition 2.3** (*optimal synchronizing sequence*). An *optimal synchronizing sequence* of length  $l \in \mathbb{N}_{2+}$  is a word  $S \in \{0, 1\}^l$  whose autodistance is maximal; in symbols,  $S \in \{0, 1\}^l$  is an optimal synchronizing sequence if and only if

$$\forall (T \in \{0, 1\}^l) \quad d(T) \leq d(S).$$

Informally, we call any word  $S \in \{0, 1\}^l$  whose autodistance is maximal or nearly maximal a *synchronizing sequence* of length  $l$ .

**Definition 2.4** (*double-optimal synchronizing sequence*). A *double-optimal synchronizing sequence* of length  $l \in \mathbb{N}_{2+}$  is a word  $S \in \{0, 1\}^l$  whose autodistance is maximal, and whose reverse autodistance is minimal among all words in  $\{0, 1\}^l$  having the maximal autodistance; in symbols,  $S \in \{0, 1\}^l$  is a double-optimal synchronizing sequence if and only if

$$\forall (T \in \{0, 1\}^l) \quad d(T) < d(S) \vee (d(T) = d(S) \wedge d'(T) \geq d'(S)).$$

Informally, any word  $S \in \{0, 1\}^l$  whose autodistance is maximal or nearly maximal and whose reverse autodistance is, among the words having the same autodistance as  $S$ , minimal or nearly minimal, will be called a *double synchronizing sequence* of length  $l$ .

**Definition 2.5** (*uniform sequence*). A *uniform sequence* is a word  $S \in \{0, 1\}^{2+}$  such that

$$d(S) = d'(S).$$

It follows from Definitions 2.1 and 2.2 above that the sequence  $S \in \{0, 1\}^{2+}$  is uniform if and only if the number  $d(S, \tau(S))$ , where  $\tau$  is a nonidentical circular permutation, does not depend on the choice of  $\tau$ .

*Examples:* The null word of any length is a uniform sequence. A word of any length containing a unique 1 and having all other digits equal to 0 is a uniform sequence.

**Definition 2.6** (*uniform optimal synchronizing sequence*). A word from  $\{0, 1\}^{2+}$  is a *uniform optimal synchronizing sequence* if it is a uniform sequence and an optimal synchronizing sequence.

Informally, any word from  $\{0, 1\}^{2+}$  which is both a uniform sequence and a synchronizing sequence will be called a *uniform synchronizing sequence*.

It follows from the definitions above that a uniform optimal synchronizing sequence is also a double-optimal synchronizing sequence.

*Example:* The word 001 is a uniform optimal synchronizing sequence. Long optimal synchronizing sequences are never trivial.

### 3. Bounds on synchronizing sequence characteristics

Theorem 3.1 establishes an upper bound on the autodistances of synchronizing sequences. Theorems 3.6 and 3.7 establish that uniform synchronizing sequences of certain forms do not exist. Theorem 3.8 states that all optimal synchronizing sequences in a certain category are uniform.

#### 3.1. An upper bound on the autodistance

**Theorem 3.1** (an upper bound on the autodistance). *For every  $l \in \mathbb{N}, l \geq 3$ , the autodistance of every word  $S \in \{0, 1\}^l$  is less than or equal to the value given in Table 1 (for  $n \in \mathbb{Z}$ ).*

Table 1

$l =  S $	$d(S)$
$4n$	$2n$
$4n + 1$	$2n$
$4n + 2$	$2n$
$4n + 3$	$2n + 2$

**Definition 3.2** ( $\text{up}(l)$ ). For every  $l \geq 3$ , the upper bound given in Table 1 is denoted as  $\text{up}(l)$ .

In order to prove Theorem 3.1, let us establish two lemmas.

**Lemma 3.3** (parity of  $d(S)$ ). *The autodistance of every word  $S \in \{0, 1\}^{2+}$  is even.*

**Proof.** By Definition 2.1, for some  $p \in \mathbb{N}$  we have  $d(S) = d(S, \tau_p(S))$ . It is, therefore, sufficient to prove that the Hamming distance between a word  $S \in \{0, 1\}^{2+}$  and any of its circular permutations is even.

Let  $T$  be a circular permutation of  $S$ . We define, for  $x, y \in \{0, 1\}$ , the four sets

$$A_{xy} = \{i \in [0 \dots |S|) \mid S[i] = x \wedge T[i] = y\},$$

which trivially have the following properties:

$$|S|_1 = |A_{10}| + |A_{11}|,$$

$$|T|_1 = |A_{01}| + |A_{11}|,$$

$$d(S, T) = |A_{01}| + |A_{10}|.$$

These equations, together with the fact that  $|S|_1 = |T|_1$ , imply

$$d(S, T) = 2|A_{01}|;$$

so,  $d(S, T)$  is even.  $\square$

**Lemma 3.4** (a weaker version of Theorem 3.1). *For  $l \geq 3$ , the autodistance of every word  $S \in \{0, 1\}^l$  is less than or equal to  $\lceil l/2 \rceil$ .*

**Proof.** Let  $S \in \{0, 1\}^l$ . We define for  $i \in [0..l)$  and  $x \in \{0, 1\}$

$$N_x[i] = |\{p \in [0..l) \mid \tau_p(S)[i] = x\}|.$$

By definition of  $\tau_p(S)$ ,

$$N_x[i] = |\{p \in [0..l) \mid S[(i+p) \bmod l] = x\}|$$

and, regardless of  $i$ ,

$$N_x[i] = |S|_x. \quad (1)$$

Let us define the *total autodistance* of  $S$ , denoted by  $K$ , as

$$K = \sum_{p=0}^{l-1} d(S, \tau_p(S)). \quad (2)$$

By definition of  $d(S, T)$ ,  $K$  satisfies:

$$\begin{aligned} K &= \sum_{p=0}^{l-1} |\{i \in [0..l) \mid S[i] \neq \tau_p(S)[i]\}| \\ &= |\{(p, i) \in [0..l)^2 \mid S[i] \neq \tau_p(S)[i]\}| \\ &= \sum_{i=0}^{l-1} |\{p \in [0..l) \mid S[i] \neq \tau_p(S)[i]\}| \\ &= \sum_{\substack{i \in [0..l) \\ S[i]=0}} N_1[i] + \sum_{\substack{i \in [0..l) \\ S[i]=1}} N_0[i] \\ &= \sum_{\substack{i \in [0..l) \\ S[i]=0}} |S|_1 + \sum_{\substack{i \in [0..l) \\ S[i]=1}} |S|_0 \quad [\text{by (1)}], \\ K &= 2|S|_0|S|_1. \end{aligned} \quad (3)$$

The autodistance of  $S$  is, by definition, the minimum of the family  $(d(S, \tau_p(S)))_{p \in \{1 \dots l\}}$ . Let us define the *average autodistance* of  $S$ , denoted by  $M$ , as the average of the same family:

$$M = \frac{\sum_{p=1}^{l-1} d(S, \tau_p(S))}{l-1}. \quad (4)$$

This definition implies that  $M \geq d(S)$ .

Equations (2) and (4) and the fact that  $d(S, \tau_0(S)) = 0$  lead to the following expression for  $M$ :

$$\begin{aligned} M &= \frac{K}{l-1}, \\ &= \frac{2|S|_0|S|_1}{l-1} \quad [\text{by (3)}]. \end{aligned} \quad (5)$$

If  $l$  is even,  $M$  is maximal for  $|S|_0 = |S|_1 = l/2$ , and we have

$$M \leq \frac{2(l/2)(l/2)}{l-1},$$

i.e.

$$M \leq \frac{l}{2} + \frac{1}{2(1-1/l)}.$$

Since  $l \geq 3$ ,

$$M < \frac{l}{2} + 1.$$

Since  $d(S) \leq M$  and  $d(S) \in \mathbf{Z}$ ,

$$d(S) \leq \frac{l}{2},$$

and the lemma holds for  $l$  even.

If  $l$  is odd,  $M$  is maximal for  $|S|_0 = (l-1)/2$  and  $|S|_1 = (l+1)/2$ . We have, therefore,

$$M \leq \frac{2(l/2 + 1/2)(l/2 - 1/2)}{l-1},$$

i.e.

$$M \leq \frac{l+1}{2}. \quad (6)$$

Then,

$$d(S) \leq \lceil l/2 \rceil,$$

and the lemma holds for  $l$  odd.  $\square$

Table 2

$l =  S $	$d(S)$
$4n$	$2n$
$4n+1$	$2n+1$
$4n+2$	$2n+1$
$4n+3$	$2n+2$

**Proof of Theorem 3.1.** Lemma 3.4 implies that, for  $l \geq 3$ , no word can have an autodistance greater than the value  $d(S)$  listed in Table 2. Lemma 3.3 says that no word can have an autodistance of the form  $2n+1$ , which makes us deduce the table in Theorem 3.1 from the one above.  $\square$

### 3.2. Nonexistence of certain uniform sequences

**Lemma 3.5** (domain of  $d'(S)$ ). *For any word  $S \in \{0, 1\}^l$ ,  $l \in \mathbb{N}_{2+}$ , the reverse autodistance of  $S$  is even and satisfies*

$$d(S) \leq d'(S) \leq l. \quad (7)$$

**Proof.** Substitution of  $d'(S)$  for  $d(S)$  in the proof of Lemma 3.3 gives the evenness of  $d'(S)$ . Relation (7) results directly from the definitions of autodistance and reverse autodistance.  $\square$

**Theorem 3.6** (nontrivial uniform sequences for  $l-1$  prime). *Let  $l \in \mathbb{N}_{2+}$  and let  $l-1$  be prime. Then, among the words  $S \in \{0, 1\}^l$ , exactly those verifying one of the conditions,*

$$|S|_0 = 0, \quad (8)$$

$$|S|_0 = 1, \quad (9)$$

$$|S|_0 = l, \quad (10)$$

$$|S|_0 = l-1, \quad (11)$$

*are uniform sequences.*

**Proof.** The reader may easily verify the fact that each of the conditions (8)–(11) implies that  $S$  is a uniform sequence.

Supposing that  $l-1$  is prime and that  $S \in \{0, 1\}^l$  is a uniform sequence, let us prove that one of the relations (8)–(11) holds. From the definitions of autodistance and reverse autodistance, we get

$$\forall (p \in [1..l]) \quad d(S) \leq d(S, \tau_p(S)) \leq d'(S),$$



which implies that  $M$ , the average autodistance of  $S$ , defined by (4), satisfies

$$d(S) \leq M \leq d'(S).$$

Since  $d(S) = d'(S)$ , we successively get

$$M = d(S),$$

$$M \in 2\mathbb{N} \quad [\text{from Lemma 3.3}],$$

$$\frac{2|S|_0|S|_1}{l-1} \in 2\mathbb{N} \quad [\text{from (5)}],$$

$$|S|_0(1 - |S|_0) \in (l-1)\mathbb{N},$$

$$|S|_0 \in (l-1)\mathbb{N} \quad \text{or} \quad (1 - |S|_0) \in (l-1)\mathbb{N} \quad [\text{since } l-1 \text{ is prime}]. \quad (12)$$

Relation (12) implies that one of the conditions (8)–(11) holds.  $\square$

**Theorem 3.7** (uniform optimal synchronizing sequences). *Let  $l \in \mathbb{N}_{2+}$ . If one of the following holds,*

(i)  $l = 4n$ , where  $n \in \mathbb{N}$  and  $\sqrt{n} \notin \mathbb{N}$ ,

(ii)  $l = 4n + 1$ , where  $n \in \mathbb{N}$  and  $\sqrt{8n+1} \notin \mathbb{N}$ ,

(iii)  $l = 4n + 2$ , where  $n \in \mathbb{N}$  and  $\sqrt{3n+1} \notin \mathbb{N}$ ,

*then no uniform sequence  $S \in \{0, 1\}^l$  will satisfy the equality  $d(S) = \text{up}(l)$ .*

**Proof.** Suppose that  $S \in \{0, 1\}^l$  is a uniform sequence with  $d(S) = d'(S) = \text{up}(l)$ . Then, reasoning as in the proof of Theorem 3.6, we can say that  $M$ , the average autodistance of  $S$ , satisfies

$$M = d(S),$$

which, by (5), translates into

$$2|S|_0(l - |S|_0) = (l-1)\text{up}(l). \quad (13)$$

If (i) holds, then  $l = 4n$ , and (13) becomes

$$|S|_0^2 - 4n|S|_0 + 4n^2 - n = 0.$$

Solving this second degree equation in  $|S|_0$ , we deduce that (13) is equivalent to

$$|S|_0 = 2n + \sqrt{n} \quad \text{or} \quad |S|_0 = 2n - \sqrt{n},$$

which is impossible since  $\sqrt{n} \notin \mathbb{N}$ .

If (ii) holds, then (13) becomes

$$|S|_0^2 - (4n+1)|S|_0 + 4n^2 = 0,$$

$$|S|_0 = \frac{1}{2}(4n+1 + \sqrt{8n+1}) \quad \text{or} \quad |S|_0 = \frac{1}{2}(4n+1 - \sqrt{8n+1}). \quad (14)$$

Recalling that the square root of a natural number is either natural or irrational, we deduce that  $\sqrt{8n+1}$  is irrational. Therefore, the alternative (14) implies that  $|S|_0$  is irrational, which is impossible.

If (iii) holds, then (13) becomes

$$\begin{aligned} |S|_0^2 - 2(2n+1)|S|_0 + (4n+1)n &= 0, \\ |S|_0 &= 2n+1 + \sqrt{3n+1} \quad \text{or} \quad |S|_0 = 2n+1 - \sqrt{3n+1}, \end{aligned} \quad (15)$$

which is impossible since  $\sqrt{3n+1} \notin \mathbb{N}$ .  $\square$

### 3.3. Uniformity of certain sequences

**Theorem 3.8** (certain sequences are uniform). *For  $l=4n+3$ ,  $n \in \mathbb{N}$ , every word from  $\{0,1\}^l$  whose autodistance is equal to  $\text{up}(l)$ , is a uniform optimal synchronizing sequence.*

Theorem 4.1 below says that sequences satisfying the hypotheses of Theorem 3.8 exist for  $l=2^n-1$ ,  $n \in \mathbb{N}_{2+}$ . In Section 5.2 (Fig. 2 and Table 3), examples of sequences are quoted for  $l=3, 7, 11, 15, 19, 23, 31, 35$ .

**Proof of Theorem 3.8.** Let  $S$  satisfy the hypotheses of the theorem. Then  $S$  is, by Theorem 3.1 and by the definition of  $\text{up}(l)$ , an optimal synchronizing sequence.

Let us prove that  $S$  is a uniform sequence. We use  $M$ , as defined by Eq. (4) in the proof of Lemma 3.4. Since  $l$  is odd, we can, as in the proof of Lemma 3.4, obtain inequality (6). This inequality and the fact that  $d(S)=(l+1)/2$  imply that  $M \leq d(S)$ . Since  $M$  is, by definition, greater than or equal to  $d(S)$ , we get

$$M = d(S).$$

The average and the minimum of the finite family of integers  $(d(S, \tau_p(S)))_{p \in [1 \dots l]}$  are then equal. All the numbers in the family are, therefore, equal and  $d'(S) = d(S)$ .  $\square$

## 4. Maximal-period linear recurring sequences

**Theorem 4.1** ( $\text{up}(l)$  is optimal for  $l=2^n-1$ ). *For every  $l$  of the form  $l=2^n-1$ ,  $n \in \mathbb{N}_{2+}$ , there exists a word  $S_n \in \{0,1\}^l$  satisfying*

$$d(S_n) = d'(S_n) = \text{up}(l). \quad (16)$$

Since this theorem is a straightforward corollary of known results, we will not quote the proof in its entirety. Instead, we only describe a way to construct the sequence  $S_n$ . The proof that this construction is correct and that the resulting  $S_n$  satisfies relation (16) is a direct consequence of well-known results from the theory of finite fields

(see e.g. [4], paragraphs 2.11, 6.32, 6.33 and 7.44). The construction itself is discussed in detail by Sarwate and Pursley ([7], Section 3).

*Construction:* Let  $\mathbf{GF}_2$  denote the Galois field of order 2 (i.e. the field composed of elements 0 and 1) and  $\mathbf{GF}_2[X]$  the ring of polynomials over  $\mathbf{GF}_2$ .

For every  $n \in \mathbb{N}_{2+}$ , there exists in  $\mathbf{GF}_2[X]$  at least one primitive polynomial of degree  $n$  (see [4], 2.11). Let us choose one such polynomial and call it  $P_n$ ; the coefficients of  $P_n$  will be called  $p_0, \dots, p_n$  (with  $p_n = 1$ ):

$$P_n(X) = p_0 + p_1 X + \dots + p_n X^n.$$

$P_n$  can be used as the characteristic polynomial to build an infinite *linear feedback sequence* of bits  $S'_n$ . To build  $S'_n$ , we arbitrarily choose its first  $n$  bits  $S'_n[0], \dots, S'_n[n-1]$ , with the only restriction that these bits may not be all equal to 0 (this gives us  $2^n - 1$  different choices of  $S'_n$ ). Then, we define the other bits of  $S'_n$  by the recurrence formula

$$0 = p_0 S'_n[i] + p_1 S'_n[i+1] + \dots + p_n S'_n[i+n] \quad (\text{for any } i \in \mathbb{N}), \quad (17)$$

which translates into

$$S'_n[i+n] = p_0 S'_n[i] + p_1 S'_n[i+1] + \dots + p_{n-1} S'_n[i+n-1] \quad (\text{for any } i \in \mathbb{N}). \quad (18)$$

The sequence  $S'_n$  is periodic and its least period is  $l = 2^n - 1$  (see [4], 6.33). We define  $S_n$  to be the left factor of  $S'_n$  of length  $l$  (therefore,  $S_n$  represents one period of  $S'_n$ ).  $S_n$  satisfies (16) (see [4], 7.44).

*Consequences of Theorem 4.1:* Theorem 4.1 implies that for all values  $l$  of the form  $2^n - 1$ , the upper bound  $\text{up}(l)$  is achieved by some word from  $\{0, 1\}^l$ . For these values of  $l$  the upper bound  $\text{up}(l)$  cannot, therefore, be improved.

The results presented in the remainder of this article imply that, in fact, the upper bound  $\text{up}(l)$  is optimal or nearly optimal for *any* length  $l$ .

## 5. Example double synchronizing sequences

### 5.1. How the examples have been bound

*Simulated annealing*, the technique used here to find double synchronizing sequences, was first described by Kirkpatrick et al. [3]. Let us describe briefly both the technique and the way in which it has been adapted to our problem.

Simulated annealing is an optimization algorithm. It provides approximate solutions to difficult problems (i.e. to problems for which finding the global optimum would involve an extremely long computing time). More precisely, for a set  $\mathcal{X}$ , on which is defined a function, called *energy*,  $\mathcal{E}: \mathcal{X} \rightarrow \mathbf{R}$ , simulated annealing will try to find an element  $x \in \mathcal{X}$  such that  $\mathcal{E}(x)$  be as low as possible.

In our case, the algorithm is run separately for each value of  $l$  and we have  $\mathcal{X} = \{0, 1\}^l$ . When searching for synchronizing sequences, we try to maximize  $d(x)$ ;

therefore,  $\mathcal{E}(x) = -d(x)$ . When searching for double synchronizing sequences, we try both to maximize  $d(x)$  and to minimize  $d'(x)$ . In this case, the choice of  $\mathcal{E}$  is not obvious; after experimentation, the author chose  $\mathcal{E}(x) = d'(x) - 3d(x)$ , although various other formulas apparently lead to identical results.

Simulated annealing requires that for every  $x \in \mathcal{X}$ , a set of neighbors  $\mathcal{N}(x)$  be defined. Intuitively,  $x$  and  $y$  are neighbors (i.e.  $y \in \mathcal{N}(x)$ ) if they are similar in a way implying that  $\mathcal{E}(x) \approx \mathcal{E}(y)$ . In our case, we consider that two words from  $\{0, 1\}^l$  are neighbors if their Hamming distance is equal to 0 or 1. For the two energy functions mentioned above, this implies that if  $y \in \mathcal{N}(x)$ , then respectively  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq 2$  or  $|\mathcal{E}(x) - \mathcal{E}(y)| \leq 8$ .

The simulated annealing algorithm is a loop composed of a high number of similar steps. In each step, the algorithm tries to update the *current solution*  $x \in \mathcal{X}$ . To do so, it randomly chooses a solution  $y \in \mathcal{N}(x)$ . Then, if  $y$  is better than  $x$  (i.e.  $\mathcal{E}(y) \leq \mathcal{E}(x)$ ),  $y$  replaces  $x$  and becomes the current solution. Otherwise (i.e. if  $\mathcal{E}(y) > \mathcal{E}(x)$ ), one of the two possibilities is randomly selected: either, with probability  $p = e^{(\mathcal{E}(x) - \mathcal{E}(y))/\theta}$ ,  $y$  replaces  $x$  and becomes the current solution, or, with probability  $1 - p$ ,  $x$  remains the current solution and  $y$  is discarded.

The current solution  $x$  present after the last step is output by the algorithm to be considered as its result.

The parameter  $\theta$  is a positive real number, called *temperature*; it decreases slowly during the computation from a problem-dependent initial value to zero. Note that for  $\theta$  very high, the algorithm reduces to random walking through the search space  $\mathcal{X}$ , regardless of the energy function (because for  $\theta$  high, always  $p \approx 1$ ); for  $\theta \approx 0$ , the algorithm descends quickly towards a local minimum of  $\mathcal{E}$ . For intermediate values of  $\theta$ , the algorithm randomly walks through  $\mathcal{X}$ , visiting more frequently elements  $x$  with  $\mathcal{E}(x)$  low.

## 5.2. What we can learn from the examples

The curve on Fig. 1 (and its magnified version, Fig. 2) shows, for each  $l \in [3..405]$ , the autodistance and the reverse autodistance of the best double synchronizing sequence found for the length  $l$  by simulated annealing. The autodistance can be compared to  $\text{up}(l)$ , also shown in the figures. Table 3 reproduces part of these results.

### 5.2.1. The autodistance

For  $3 \leq l \leq 42$ , the autodistance of the examples is, with the exceptions of  $l = 27$  and  $l = 39$ , equal to  $\text{up}(l)$ . For the particular cases of  $l = 27$  and  $l = 39$ , exhaustive searches showed that there are no synchronizing sequences with autodistance equal to  $\text{up}(l)$ <sup>1</sup>; the examples found for these two values of  $l$  are, therefore, optimal.

<sup>1</sup> For  $l = 39$ , the exhaustive search was performed by Mark Shand [8] using a carefully optimized search program.

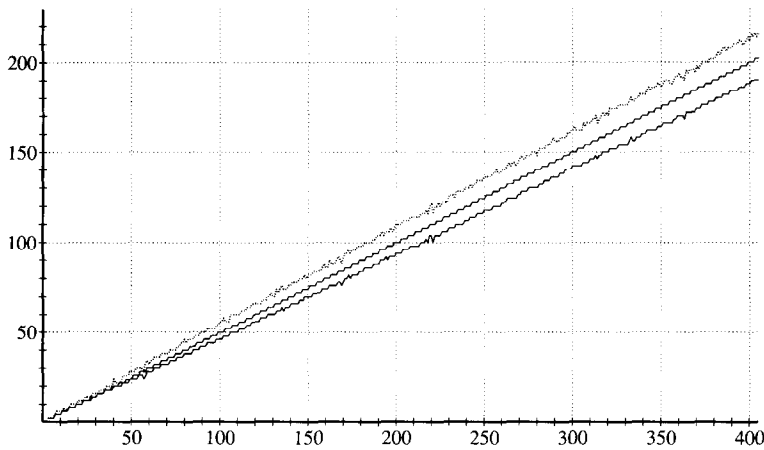


Fig. 1. Autodistance and reverse autodistance of example sequences as a function of their lengths  $l$ . The lower line shows the autodistance of the best double synchronizing sequence found by simulated annealing for each length. The upper, dotted line shows the reverse autodistance of the same sequences. The middle, perfectly regular line shows  $up(l)$ .

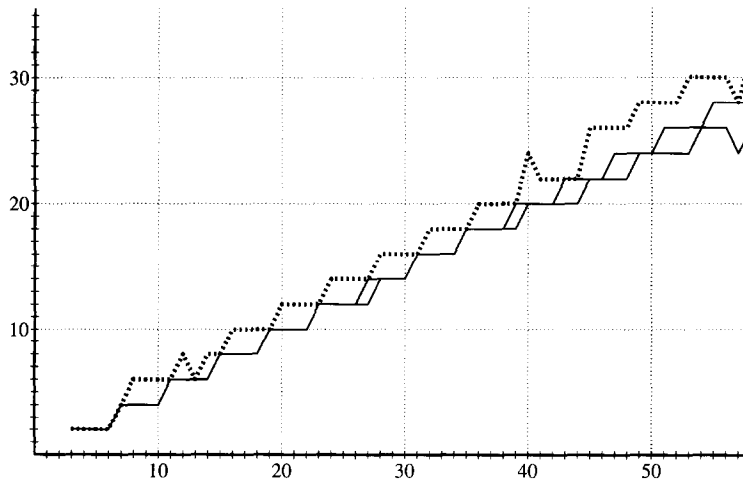


Fig. 2. A fragment of the curves from Fig. 1, magnified.

We are thus certain that, for  $l \leq 42$  (as well as for  $l = 45, 46, 49, 50, 54$ ; see Fig. 2), the simulated annealing program actually found optimal synchronizing sequences. For these values, with the exceptions of  $l = 27$  and  $l = 39$ , the upper bound of Theorem 3.1 is exact. For  $l = 27$  and  $l = 39$ , the maximal autodistance is less than  $up(l)$ , and Theorem 3.1 could be improved to take this fact into account.

According to Theorem 4.1, for lengths of the form  $l = 2^n - 1$ , some sequences achieve the upper bound  $up(l)$ . Therefore, for  $l = 63, 127, 255$ , the simulated annealing program found only suboptimal synchronizing sequences.

Table 3  
Examples of synchronizing sequences

$l =  S $	$d(S)$	$d'(S)$	$S$
3	2	2	100
4	2	2	0100
5	2	2	01000
6	2	2	000100
7	4	4	1110100
8	4	6	11100010
9	4	6	110000010
10	4	6	0000010110
11	6	6	10001001011
12	6	8	111001100101
13	6	6	1000000101001
14	6	8	11100100010000
15	8	8	000100110101111
16	8	10	1101110000011010
17	8	10	11001101101010001
18	8	10	110010110010000101
19	10	10	1001111010100001100
20	10	12	01000011011011000101
21	10	12	011110000100101110110
22	10	12	0100001010001001111011
23	12	12	00000101001100110101111
24	12	14	100011110110110000010101
25	12	14	1011000110000000101110100
26	12	14	10010100111110001000100010
27	12	14	110100010111001100000000010
28	14	16	0111001111110100100110101000
29	14	16	00000101100111111001010011101
30	14	16	1110011001011101010111000111111
31	16	16	1111011010011000001110010001010
32	16	18	00010001011001000111011010111100
33	16	18	100100111000111011101000010000101
34	16	18	1010001111011010010011001100000010
35	18	18	00000111000101101100101011110110001
36	18	20	100010011110111100001011010001011000
37	18	20	0011011010111010001100001000110111101
38	18	20	01010001000000011001111000110110100001
39	18	20	010010110101110011100000011101000100010
40	20	24	0001000011101000110100110011010110111
41	20	22	00011101011111000001001010000100110110001
42	20	22	111111010000001000100110001010010010111000
43	20	22	1110110001010111100100111101001110010111011
44	20	22	11110110100111111100111110101010011001001110
45	22	26	001000110001101000101110001011010011011111101
46	22	26	1011010110111010010001000111110001110010010111
47	22	26	01111010101000101101011000001100010011110011011
48	22	26	01101100011000101010111001001011110100000011000
49	24	28	010000110101110111110110000011100110110000101010
50	24	28	11000010110111001010011001101110101110000100000110

For  $l=43, 44, 48$ , by systematically searching through a significant fraction of  $\{0,1\}^l$ , Mark Shand [8] found words achieving  $\text{up}(l)$ ; the best examples found by simulated annealing for these values of  $l$  are, therefore, nonoptimal.

For all values of  $l$  not mentioned above, we do not know whether the synchronizing sequences found using simulated annealing are optimal; nor do we know whether  $\text{up}(l)$  is the exact upper bound for these values. Unlike for  $l \leq 44$ , the exhaustive search, which costs  $O(2^l)$  in time, cannot be applied to answer these questions.

### 5.2.2. The reverse autodistance of optimal synchronizing sequences

Lemma 3.5 and Theorem 3.7 imply that the examples found for  $l \in \{3 \dots 15, 17 \dots 21, 23, 24, 26, 28 \dots 33, 35, 37, 42\}$  are double-optimal synchronizing sequences.

As indicated in Section 5.2.1, for  $l=27$  there are no words  $S \in \{0,1\}^l$  with  $d(S) = \text{up}(l)$ ; a computation analogous to these in the proof of Theorem 3.7 shows that there is also no word of this length with  $d(S) = d'(S) = \text{up}(l) - 2$ . Therefore, the corresponding example is a double-optimal synchronizing sequence.

For  $l=16, 22, 25$ , exhaustive searches showed that there is no word  $S \in \{0,1\}^l$  satisfying  $d(S) = d'(S) = \text{up}(l)$ ; the corresponding examples are, therefore, double-optimal synchronizing sequences.

For  $l \in \{34, 36, 38, 40, 41, 45, 46, 49, 50, 54\}$ , the examples found are optimal synchronizing sequences, but the author has not been able to establish whether they are double-optimal.

## 6. Double synchronizing sequences of length $l \rightarrow +\infty$

### 6.1. The result

**Theorem 6.1** (double synchronizing sequences for large  $l$ ). *Let  $\alpha \in \mathbf{R}$ ,  $0 < \alpha < 1$ . There exists a function  $\varepsilon: \mathbf{N}_{2+} \rightarrow \mathbf{R}_+$  such that  $\lim_{+\infty} \varepsilon = 0$  and that for every  $l \in \mathbf{N}$ ,  $l \geq 3$ , there are at least  $\alpha 2^l$  distinct words  $S \in \{0,1\}^l$  satisfying*

$$\text{up}(l) - l\varepsilon(l) \leq d(S) \leq d'(S) \leq \text{up}(l) + l\varepsilon(l).$$

### 6.2. How the proof is organized

The proof of Theorem 6.1 is long. Let us summarize it here.

Section 6.3 states two capital lemmas from which the theorem directly results.

Section 6.4 defines several notational conventions.

Section 6.5 contains auxiliary lemmas, which recall generally known mathematical facts.

Sections 6.6–6.9 contain the proof of the first capital lemma.

In Section 6.6, we choose a function  $\varepsilon$  which, as we will prove, satisfies both capital lemmas (and, thus, the theorem). We define then the set  $E \subset \{0,1\}^l$  of words whose autodistance is less than  $\text{up}(l) - \varepsilon(l)l$ , and we represent it as equal to the union of a family of sets called  $E_{p,D}$ .

Then, in Sections 6.7 and 6.8, we establish intermediate results which will enable us to estimate the cardinals of the sets  $E_{p,D}$ . Finally, in Section 6.9, we use these results to prove that  $|E| \leq ((l-\alpha)/2)2^l$ , from what the first capital lemma results.

In Section 6.10, rather than fully describing the proof of the second capital lemma, we simply indicate in which ways it differs from the proof of the first capital lemma.

### 6.3. The two capital lemmas

Theorem 6.1 follows in a straightforward way from the two following lemmas.

**Capital Lemma 6.2** (autodistance for high  $l$ ). *Let  $\alpha \in \mathbf{R}$ ,  $0 < \alpha < 1$ . There exists a function  $\varepsilon: \mathbf{N}_{2+} \rightarrow \mathbf{R}_+$  such that  $\lim_{+\infty} \varepsilon = 0$  and for every  $l \in \mathbf{N}$ ,  $l \geq 3$ , there are at most  $((1-\alpha)/2)2^l$  distinct words  $S \in \{0, 1\}^l$  such that*

$$d(S) < \text{up}(l) - l\varepsilon(l).$$

**Capital Lemma 6.3** (reverse autodistance for high  $l$ ). *Let  $\alpha \in \mathbf{R}$ ,  $0 < \alpha < 1$ . There exists a function  $\varepsilon: \mathbf{N}_{2+} \rightarrow \mathbf{R}_+$  such that  $\lim_{+\infty} \varepsilon = 0$  and for every  $l \in \mathbf{N}$ ,  $l \geq 3$ , there are at most  $((1-\alpha)/2)2^l$  distinct words  $S \in \{0, 1\}^l$  such that*

$$\text{up}(l) + l\varepsilon(l) < d'(S).$$

### 6.4. Conventions

We make, for the whole proof, the following assumptions about the numbers  $l, p, a, b$  and  $\mu$  and about the sets  $D$  and  $P$ :

$$\begin{aligned} l &\in \mathbf{N}_{2+}, & 3 &\leq l, \\ p &\in \mathbf{Z}, & 1 &\leq p \leq l/2, \\ a &\in \mathbf{N}, & 1 &\leq a, \\ b &\in \mathbf{N}, & 2 &\leq b, \\ \mu &\in \mathbf{R}, & 0 &< \mu < 1/2, \\ D &\subset [0 \dots l], \\ P &\subset \mathbf{Z}, & P &\text{ is a finite set.} \end{aligned}$$

These assumptions are valid in lemmas and auxiliary definitions which are part of the proof. They will not be recalled there. For instance, Example Lemma 6.4 will be abbreviated to Example Lemma 6.5.

**Example Lemma 6.4.** *For every  $\mu \in \mathbf{R}$  such that  $0 < \mu < 1/2$  and for every  $n \in \mathbf{Z}$ ,  $\mu \neq n$ .*

**Example Lemma 6.5.** *For every  $n \in \mathbf{Z}$ ,  $\mu \neq n$ .*



## 6.5. Auxiliary lemmas

**Lemma 6.6** (approximation of  $\binom{n}{d}$ ). For every  $n, d \in \mathbf{N}$ ,

$$d \leq (1/2 - \mu)n - 1 \text{ implies } \binom{n}{d} \leq 2^n e^{-\mu^3 n}.$$

**Proof** (outline). Let us define  $q = \lfloor (1/2 - \mu/2)n \rfloor$ . Using the well-known equality  $\binom{n}{r} = n!/(r!(n-r)!)$ , we can then state the following:

$$\begin{aligned} \forall (r \in [d \dots q]) \quad \binom{n}{r} &\leq \frac{1-\mu}{1+\mu} \binom{n}{r+1}, \\ \binom{n}{d} &\leq \left( \frac{1-\mu}{1+\mu} \right)^{\mu n/2} \binom{n}{q}, \\ \binom{n}{d} &\leq \left( \frac{1-\mu}{1+\mu} \right)^{\mu n/2} 2^n, \\ \binom{n}{d} &\leq e^{-\mu^3 n} 2^n. \quad \square \end{aligned}$$

**Auxiliary Definition 6.7** (families  $\mathcal{F}_i$ ). For  $i \in [0 \dots l \sqcap p)$  and  $x \in [0 \dots l/l \sqcap p)$ , we define the numbers

$$F_{ix} = (xp + i) \bmod l$$

which form the families

$$\mathcal{F}_i = (F_{ix})_{0 \leq x < l/l \sqcap p}.$$

The numbers  $F_{ix}$  and the families  $\mathcal{F}_i$  depend on the numbers  $l$  and  $p$  but, for simplicity,  $l$  and  $p$  do not appear as indices in their notation.

**Lemma 6.8** (fundamental property of  $\mathcal{F}_i$ ). For every  $i \in [0 \dots l \sqcap p)$ , the family  $\mathcal{F}_i$  contains exactly once every element of the set  $A_i = ((l \sqcap p)\mathbf{Z} + i) \cap [0 \dots l)$  and contains only elements of this set.

**Proof** (outline). We call  $\text{Im}(\mathcal{F}_i)$  the image of the family  $\mathcal{F}_i$ , namely,

$$\text{Im}(\mathcal{F}_i) = \{F_{ix} \mid 0 \leq x < l/l \sqcap p\}.$$

For every  $x, y \in [0 \dots l/l \sqcap p)$ , the equation  $F_{ix} = F_{iy}$  is equivalent to

$$\exists (k \in \mathbf{Z}), (x - y) \frac{p}{l \sqcap p} = k \frac{l}{l \sqcap p},$$

which, thanks to the Gauss theorem [6], implies that  $x - y \in (l/l \cap p)\mathbf{Z}$ . Since  $-l/l \cap p < x - y < l/l \cap p$ , we get  $x = y$ . All the elements of the family  $\mathcal{F}_i$  are therefore distinct and the family contains every element of  $A_i$  at most once.

Since all the elements of  $\mathcal{F}_i$  are distinct, the set  $\text{Im}(\mathcal{F}_i)$  contains  $l/l \cap p$  elements;  $A_i$  and  $\text{Im}(\mathcal{F}_i)$  have, therefore, the same number of elements. Since, as the reader may easily verify,  $\text{Im}(\mathcal{F}_i) \subset A_i$ , we get  $\text{Im}(\mathcal{F}_i) = A_i$ . The family  $\mathcal{F}_i$  contains then each element of  $A_i$  at least once and contains no elements from outside  $A_i$ .  $\square$

**Lemma 6.9** (*parity of the cardinal*). *If  $A$  and  $B$  are finite sets,  $|A \triangle B|$  has the same parity as  $|A| + |B|$ . In other words,*

$$|A \triangle B| \equiv |A| + |B| \pmod{2}.$$

The proof is left to the reader.

#### 6.6. The sets $E_{p,D}$

Let  $\alpha$  be defined as in Capital Lemma 6.2. We define then

$$\begin{aligned} \mu'(l) &= \left( \frac{1}{\ln l} + \frac{2}{l} \ln l \ln \frac{2l^2}{1-\alpha} \right)^{1/3}, \\ \varepsilon'(l) &= \mu'(l) + \frac{2 \ln l}{l} + \frac{1}{l}, \\ \varepsilon(l) &= \begin{cases} \varepsilon'(l) & \text{if } \varepsilon'(l) < 1/2 \text{ and } \mu'(l) < 1/2, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

The functions  $\mu'$ ,  $\varepsilon'$  and  $\varepsilon$  are then strictly positive, and satisfy

$$\lim_{+\infty} \mu' = 0,$$

$$\lim_{+\infty} \varepsilon' = 0,$$

$$\lim_{+\infty} \varepsilon = 0.$$

(The easy, computational proofs of these facts are not reproduced here.)

To prove Capital Lemma 6.2, it is now sufficient to establish, for every  $l$ , the property that there are at most  $((1-\alpha)/2)2^l$  distinct words  $S \in \{0, 1\}^l$  such that  $d(S) < \text{up}(l) - l\varepsilon(l)$ .

For  $l$  such that  $\varepsilon'(l) \geq 1/2$  or  $\mu'(l) \geq 1/2$ , we have  $\varepsilon(l) = 1$  and the property trivially holds. We suppose therefore, for the rest of the proof, that  $\varepsilon'(l) < 1/2$  and that  $\mu'(l) < 1/2$ , and we establish the property in this case.

Define

$$\delta = \text{up}(l) - l\varepsilon(l), \tag{19}$$

$$E = \{S \in \{0, 1\}^l \mid d(S) < \delta\}. \tag{20}$$

The property to be proved can then be expressed by the relation

$$|E| \leq \frac{1-\alpha}{2} 2^l. \quad (21)$$

By Definition 2.1, Eq. (20) can be rewritten as

$$E = \{S \in \{0, 1\}^l \mid \exists (q \in [1 \dots l]) \, d(S, \tau_q(S)) < \delta\}. \quad (22)$$

From the definition of the Hamming distance, it is easy to show that for every  $q \in \mathbb{Z}$  and every  $S \in \{0, 1\}^l$ ,

$$d(S, \tau_q(S)) = d(S, \tau_{l-q}(S))$$

and that (22) is equivalent to

$$E = \{S \in \{0, 1\}^l \mid \exists (p \in [1 \dots \lfloor l/2 \rfloor]) \, d(S, \tau_p(S)) < \delta\}. \quad (23)$$

We then define

$$E_p = \{S \in \{0, 1\}^l \mid d(S, \tau_p(S)) < \delta\}. \quad (24)$$

Relation (23) can then be rewritten as

$$E = \bigcup_{p=1}^{\lfloor l/2 \rfloor} E_p. \quad (25)$$

Let us define, for  $S \in \{0, 1\}^l$ , the *set of differences*  $D_{S,p}$ :

$$D_{S,p} = \{i \in [0 \dots l] \mid S[i] \neq \tau_p(S)[i]\}, \quad (26)$$

$$D_{S,p} = \{i \in [0 \dots l] \mid S[i] \neq S[(i+p) \bmod l]\}, \quad (27)$$

and, for any  $D$ , let

$$E_{p,D} = \{S \in \{0, 1\}^l \mid D_{S,p} = D\}. \quad (28)$$

Then (24) may be rewritten as

$$E_p = \bigcup_{|D| < \delta} E_{p,D}. \quad (29)$$

From Eqs. (25) and (29), we can deduce that

$$|E| \leq \sum_{p=1}^{\lfloor l/2 \rfloor} \sum_{|D| < \delta} |E_{p,D}|. \quad (30)$$

The rest of this proof consists in bounding the number of terms in this sum and in estimating  $|E_{p,D}|$  as a function of  $l, p$  and  $D$ .

### 6.7. More auxiliary lemmas

**Auxiliary Definition 6.10** (functions  $\varphi[i]$  and expression  $f(D, i, j)$ ). For  $i \in \mathbb{N}$ , let us define the functions  $\varphi, \varphi[i]: \{0, 1\} \rightarrow \{0, 1\}$ :

$$\varphi(x) = 1 - x,$$

$$\varphi[0](x) = x,$$

$$\varphi[i+1](x) = \varphi[i] \circ \varphi(x).$$

For  $i \in [0..l \sqcap p)$  and  $j \in [0..l/l \sqcap p]$ , we define

$$f(D, i, j) = |(F_{ix})_{0 \leq x < j}|_D.$$

The expression  $f(D, i, j)$  depends on  $l$  and  $p$ , which, for simplicity, do not appear there as indices.

**Lemma 6.11** (relation between  $S[i]$ ,  $S[j]$  and  $D_{S,p}$ ). Let  $S \in E_{p,D}$ . Then, for  $0 \leq i < l \sqcap p$  and  $0 \leq j \leq l/l \sqcap p$ , we have

$$S[(i + pj) \bmod l] = \varphi[f(D, i, j)](S[i]).$$

**Proof.** First, observe that for  $n$  even,  $\varphi[n](x) = x$  and for  $n$  odd,  $\varphi[n](x) = 1 - x$ .

We will prove the lemma by induction on  $j$ ; the verification that the lemma holds for  $j=0$  is left to the reader.

Let us assume the lemma to be true for  $j$  (with  $0 \leq j < l/l \sqcap p$ ) and prove it for  $j+1$ . Under the lemma's hypotheses, the fact that  $S \in E_{p,D}$  (which implies  $D = D_{S,p}$ ) and relation (27), let us state the following:

$$\text{if } (i + pj) \bmod l \in D, \quad S[(i + p(j+1)) \bmod l] = 1 - S[(i + pj) \bmod l],$$

$$\text{otherwise, } S[(i + p(j+1)) \bmod l] = S[(i + pj) \bmod l],$$

which may be expressed as follows:

$$\begin{aligned} S[(i + p(j+1)) \bmod l] &= \varphi[|D \cap \{(i + pj) \bmod l\}|](S[(i + pj) \bmod l]) \\ &= \varphi[f(D, i, j+1) - f(D, i, j)](S[(i + pj) \bmod l]) \\ &= \varphi[f(D, i, j+1) - f(D, i, j)](\varphi[f(D, i, j)](S[i])) \\ S[(i + p(j+1)) \bmod l] &= \varphi[f(D, i, j+1)](S[i]). \quad \square \end{aligned}$$

**Lemma 6.12** (some  $E_{p,D}$  are empty). If, for some  $i \in [0..l \sqcap p)$ , the number  $|((l \sqcap p)\mathbb{Z} + i) \cap D|$  is odd, then  $E_{p,D} = \emptyset$ .

**Proof.** Let  $i \in [0..l \sqcap p)$  and let  $|((l \sqcap p)\mathbf{Z} + i) \cap D|$  be odd. By Lemma 6.8, we get

$$\begin{aligned} |\mathcal{F}_i|_D &= |(((l \sqcap p)\mathbf{Z} + i) \cap [0..l)) \cap D| \\ &= |((l \sqcap p)\mathbf{Z} + i) \cap D|; \end{aligned}$$

$|\mathcal{F}_i|_D$  is then odd. Applying Lemma 6.11, for every  $S \in E_{p,D}$  we get then

$$S[i] = S \left[ \left( i + p \frac{l}{l \sqcap p} \right) \bmod l \right],$$

$$S[i] = \varphi \left[ f \left( D, i, \frac{l}{l \sqcap p} \right) \right] (S[i]),$$

$$S[i] = \varphi [|\mathcal{F}_i|_D] (S[i]),$$

$$S[i] = 1 - S[i] \quad (\text{since } |\mathcal{F}_i|_D \text{ is odd}),$$

which is impossible. Therefore,  $S \in E_{p,D}$  is true for no  $S$  and  $E_{p,D} = \emptyset$ .  $\square$

**Lemma 6.13** ( $E_{p,D}$  has at most  $2^{l \sqcap p}$  members). *For every  $S' \in \{0, 1\}^{l \sqcap p}$ , there exists at most one  $S$  such that  $S \in E_{p,D}$  and the leftmost factor of  $S$  of length  $l \sqcap p$  is equal to  $S'$ .*

**Proof.** Let  $S \in E_{p,D}$  and  $k \in [0..l)$ . Let  $i$  be the remainder in the division of  $k$  by  $l \sqcap p$ . Since  $0 \leq i < l \sqcap p$  and  $k \in ((l \sqcap p)\mathbf{Z} + i) \cap [0..l)$ , Lemma 6.8 implies that for some  $j \in [0..l/l \sqcap p)$ , we have  $k = (i + pj) \bmod l$ . We can then apply Lemma 6.11 to get

$$S[k] = \varphi [f(D, i, j)] (S[i]).$$

This formula shows that every bit in  $S$  can be determined as a function of  $l, p, D$  and one of the  $l \sqcap p$  leftmost bits of  $S$ . Therefore, for any given values of  $l, p$  and  $D$ , the left factor of  $S$  of length  $l \sqcap p$  uniquely determines  $S$ .  $\square$

6.8. The sets  $\mathcal{Q}_{d,p}$

For any  $d \in \mathbb{N}$ , let us define

$$\mathcal{Q}_{d,p} = \{D \mid |D| < d \wedge E_{p,D} \neq \emptyset\} \quad (31)$$

(the set  $\mathcal{Q}_{d,p}$  depends on  $l$ , but for simplicity  $l$  will not appear as an index in its notation).

We can rewrite Eq. (30) as follows:

$$|E| \leq \sum_{p=1}^{\lfloor l/2 \rfloor} \sum_{D \in \mathcal{Q}_{d,p}} |E_{p,D}|. \quad (32)$$

By Lemma 6.13, for every  $D \in \mathcal{D}_{\delta, p}$ ,  $|E_{p, D}| \leq 2^{l \cap p}$  and Eq. (32) implies that

$$|E| \leq \sum_{p=1}^{\lfloor l/2 \rfloor} |\mathcal{D}_{\delta, p}| 2^{l \cap p}. \quad (33)$$

Let us find two different (and both useful) upper bounds on  $|\mathcal{D}_{\delta, p}|$ .

Since  $\mathcal{D}_{\delta, p}$  is only composed of subsets of  $[0..l)$  containing less than  $\delta$  elements, we get

$$|\mathcal{D}_{\delta, p}| \leq \sum_{0 \leq x < \delta} \binom{l}{x}, \quad (34)$$

and we can easily verify that the hypotheses of Lemma 6.6 hold (for  $\mu = \mu'(l)$ ); in this way we get the first upper bound on  $\mathcal{D}_{\delta, p}$ :

$$|\mathcal{D}_{\delta, p}| \leq l e^{-\mu'(l)^3 l} 2^l. \quad (35)$$

Let us compute the second upper bound on  $\mathcal{D}_{\delta, p}$ . To simplify notation, we define the two intervals  $I$  and  $J$ :

$$I = [0..a(b-1)),$$

$$J = [a(b-1)..ab).$$

**Auxiliary Definition 6.14** (sets  $\mathcal{D}'_{d, a, b, p}$ ). For every  $d \in \mathbb{N}$ , let

$$\mathcal{D}'_{d, a, b, p}$$

denote the set of sets  $D' \subset [0..ab)$  such that  $|D'| < d$  and, for every  $i \in [0..a)$ ,  $|D' \cap (a\mathbb{Z} + i)| + |P \cap (a\mathbb{Z} + i)|$  is even.

**Lemma 6.15.** For every  $d \in \mathbb{N}$ ,

$$|\mathcal{D}'_{d, a, b, p}| \leq 2^{a(b-1)}. \quad (36)$$

**Proof.** Since the set  $I$  has  $a(b-1)$  elements, there are at most  $2^{a(b-1)}$  possible sets of the form  $D' \cap I$ . To prove the lemma, it will, therefore, suffice to establish that for fixed  $d, a, b$  and  $P$ , and under the condition that  $D' \in \mathcal{D}'_{d, a, b, p}$ , the set  $D' \cap I$  uniquely determines  $D'$ .

For every  $i$ ,  $D' \cap (a\mathbb{Z} + i)$  is the disjoint union of  $D' \cap I \cap (a\mathbb{Z} + i)$  and  $D' \cap J \cap (a\mathbb{Z} + i)$ ; therefore,

$$|D' \cap (a\mathbb{Z} + i)| = |D' \cap I \cap (a\mathbb{Z} + i)| + |D' \cap J \cap (a\mathbb{Z} + i)|.$$

The number

$$|D' \cap I \cap (a\mathbb{Z} + i)| + |D' \cap J \cap (a\mathbb{Z} + i)| + |P \cap (a\mathbb{Z} + i)|$$

is, therefore, even. The parity of  $|D' \cap J \cap (a\mathbb{Z} + i)|$  is, hence, determined by  $D' \cap I$ .

On the other hand, we have

$$J \cap (a\mathbb{Z} + i) = \{i + a(b-1)\}.$$

Therefore,

$$|D' \cap J \cap (a\mathbb{Z} + i)| \text{ even implies } D' \cap J \cap (a\mathbb{Z} + i) = \emptyset,$$

$$|D' \cap J \cap (a\mathbb{Z} + i)| \text{ odd implies } D' \cap J \cap (a\mathbb{Z} + i) = \{i + a(b-1)\}.$$

In this way,  $D' \cap I$  uniquely determines  $D' \cap J \cap (a\mathbb{Z} + i)$  for every  $i$ . The (easy to verify) equality

$$D' = (D' \cap I) \cup \bigcup_{i=0}^{a-1} (D' \cap J \cap (a\mathbb{Z} + i))$$

implies then that  $D' \cap I$  uniquely determines  $D'$ .  $\square$

**Lemma 6.16.** *For every  $d$  such that  $0 \leq d \leq (1/2 - \mu)ab - b$ ,*

$$|\mathcal{D}'_{d,a,b,P}| \leq a e^{-\mu^3 a} 2^{ab+b-a}. \quad (37)$$

**Proof.** For every value of  $a$ , we will prove the lemma by induction on  $b$ .

First, we need to verify the lemma for  $b=2$ . This verification, when fully described, is extremely long. For this reason, we will omit here numerous computational details.

For any fixed  $d, a$  and  $P$  satisfying lemma's hypotheses and for  $b=2$ , we consider  $D'$  as a variable satisfying  $D' \in \mathcal{D}'_{d,a,2,P}$  and we estimate the number of values that  $D'$  can take (this number is obviously equal to  $|\mathcal{D}'_{d,a,2,P}|$ ).

We define the sets  $U$  and  $V$ :

$$U = \{i \in I \mid |P \cap (a\mathbb{Z} + i)| \in 2\mathbb{Z}\},$$

$$V = \{i \in I \mid |P \cap (a\mathbb{Z} + i)| \notin 2\mathbb{Z}\}.$$

It is easy to see that if  $|V| > d$ , then  $\mathcal{D}'_{d,a,2,P} = \emptyset$  and the lemma holds. We suppose, therefore, that  $|V| \leq d$  and verify the lemma in this case only.

Let us quote the following, easy to establish, relations:

$$U \cup V = [0 \dots a),$$

$$U \cap V = \emptyset,$$

$$|U| + |V| = a,$$

$$U + a \subset J,$$

$$V + a \subset J.$$

Let  $i \in V$ . The cardinal of  $D' \cap (a\mathbf{Z} + i)$  is then odd and, since  $D' \cap (a\mathbf{Z} + i) \subset \{i, a + i\}$ , we get

$$\begin{aligned} a + i \in D' &\Leftrightarrow i \notin D', \\ i \in D' - a &\Leftrightarrow i \notin D'. \end{aligned} \quad (38)$$

From here, we can deduce that

$$\begin{aligned} V \cap (D' - a) &= V - D', \\ (V + a) \cap D' &= (V - D') + a. \end{aligned} \quad (39)$$

Therefore,  $V \cap D'$  uniquely determines  $(V + a) \cap D'$ . By remarking that  $V \cap D'$  can take at most  $2^{|V|}$  different values, we conclude that  $(V \cup (V + a)) \cap D'$  can only take  $2^{|V|}$  different values.

From relation (39) we get

$$|(V + a) \cap D'| + |V \cap D'| = |V|;$$

$V$  and  $V + a$  being disjoint, we conclude that  $|(V \cup (V + a)) \cap D'| = |V|$ . Since the sets  $V \cup (V + a)$  and  $U \cup (U + a)$  are disjoint, we finally get

$$\begin{aligned} |(U \cup (U + a)) \cap D'| + |(V \cup (V + a)) \cap D'| &< d, \\ |(U \cup (U + a)) \cap D'| &< d - |V|. \end{aligned} \quad (40)$$

A relation concerning  $U$  and analogous to (39) can be established:

$$(U + a) \cap D' = (U \cap D') + a \quad (41)$$

and can be used to conclude that  $U \cap D'$  uniquely determines  $(U + a) \cap D'$ .

Relation (41), together with the fact that  $U$  and  $U + a$  are disjoint, leads to the conclusion that

$$\begin{aligned} |U \cap D'| &= |(U + a) \cap D'| \\ &= \frac{1}{2} |(U \cup (U + a)) \cap D'|, \\ |U \cap D'| &< \frac{d - |V|}{2} \quad [\text{by (40)}]. \end{aligned}$$

Since  $U \cap D'$  is a set containing less than  $(d - |V|)/2$  elements chosen among the  $a - |V|$  elements of  $U$ , it can take at most

$$\sum_{0 \leq k < (d - |V|)/2} \binom{a - |V|}{k}$$

different values; the same is true concerning  $(U \cup (U + a)) \cap D'$  (since this set is determined in a unique way by  $U \cap D'$ ).



From the fact that

$$D' = ((U \cup (U + a)) \cap D') \cup ((V \cup (V + a)) \cap D'),$$

we finally deduce that  $D'$  can take no more than

$$\sum_{0 \leq k < (d - |V|)/2} \binom{a - |V|}{k} 2^{|V|}$$

different values; then

$$|\mathcal{D}'_{d,a,2,P}| \leq a \binom{a - |V|}{\lfloor (d - |V|)/2 \rfloor} 2^{|V|}. \quad (42)$$

We can verify the following relations (remember that  $|V| \leq d$ ):

$$0 \leq \frac{d - |V|}{2} \leq \left( \frac{1}{2} - \mu \frac{a}{a - |V|} \right) (a - |V|) - 1,$$

$$0 < \mu \frac{a}{a - |V|} < \frac{1}{2},$$

which, together with (42), enable us to use Lemma 6.6 and obtain

$$|\mathcal{D}'_{d,a,2,P}| \leq a e^{-[\mu a(a - |V|)]^3 (a - |V|)} 2^{a - |V|} 2^{|V|},$$

from which we deduce that (37) holds and, thus, end the verification for  $b = 2$ .

Now, we suppose that  $b \geq 3$  and that the lemma holds for  $b' = b - 1$ . Supposing that  $a, d, \mu$  and  $P$  satisfy the lemma's hypotheses, let us establish relation (37). Let  $D' \in \mathcal{D}'_{d,a,b,P}$ . We can split  $D'$  into the union of two disjoint subsets  $D_1$  and  $Q$ :

$$D_1 = D' \cap I,$$

$$Q = D' \cap J.$$

By definition of  $\mathcal{D}'_{d,a,b,P}$ , for every  $i \in [0..a)$  we have

$$|D' \cap (a\mathbf{Z} + i)| + |P \cap (a\mathbf{Z} + i)| \in 2\mathbf{N};$$

this can be rewritten as

$$|D_1 \cap (a\mathbf{Z} + i)| + |Q \cap (a\mathbf{Z} + i)| + |P \cap (a\mathbf{Z} + i)| \in 2\mathbf{N}$$

and, by Lemma 6.9,

$$|D_1 \cap (a\mathbf{Z} + i)| + |(P \triangle Q) \cap (a\mathbf{Z} + i)| \in 2\mathbf{N}. \quad (43)$$

The fact that  $D_1 \subset I$  and that  $|D_1| + |Q| = |D'|$ , together with relation (43), enable us to state that

$$D_1 \in \mathcal{D}'_{d - |Q|, a, b - 1, P \triangle Q}.$$

We have, therefore, established that every  $D' \in \mathcal{D}'_{d,a,b,P}$  is the union of some  $Q \subset J$  and some  $D_1 \in \mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}$ . Then,

$$\begin{aligned} \mathcal{D}'_{d,a,b,P} &\subset \bigcup_{Q \subset J} \{D_1 \cup Q \mid D_1 \in \mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}\}, \\ |\mathcal{D}'_{d,a,b,P}| &\leq \sum_{Q \subset J} |\mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}|. \end{aligned} \quad (44)$$

Let us split the sum (44) into two terms  $X$  and  $Y$ :

$$\begin{aligned} |\mathcal{D}'_{d,a,b,P}| &\leq X + Y, \\ X &= \sum_{\substack{Q \subset J \\ |Q| < (1/2 - \mu)a - 1}} |\mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}|, \\ Y &= \sum_{\substack{Q \subset J \\ |Q| \geq (1/2 - \mu)a - 1}} |\mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}|. \end{aligned}$$

The sum  $X$  is indexed by subsets of  $J$  having less than  $(1/2 - \mu)a - 1$  elements. Lemma 6.6 implies then that the number of terms in the sum is less than or equal to

$$\begin{aligned} &\sum_{0 \leq i < (1/2 - \mu)a - 1} \binom{a}{i} \\ &\leq ae^{-\mu^3 a} 2^a. \end{aligned}$$

From Lemma 6.15, we deduce that each term in  $X$  is less than or equal to  $2^{ab-2a}$ ; therefore,

$$X \leq ae^{-\mu^3 a} 2^{ab-a}. \quad (45)$$

The sum  $Y$ , being indexed by subsets of  $J$ , contains at most  $2^a$  terms. Each term is of the form

$$|\mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}|$$

where

$$d - |Q| \leq (1/2 - \mu)a(b-1) - (b-1).$$

After straightforward verifications, the induction hypothesis (Lemma 6.16 applied for  $b-1$ ) may be applied to give

$$|\mathcal{D}'_{d-|Q|,a,b-1,P \triangle Q}| \leq ae^{-\mu^3 a} 2^{a(b-1) + (b-1) - a}.$$

Therefore,

$$Y \leq ae^{-\mu^3 a} 2^{ab + (b-1) - a}, \quad (46)$$

and

$$\begin{aligned} |\mathcal{D}'_{d,a,b,P}| &\leq ae^{-\mu^3 a} 2^{ab-a} + ae^{-\mu^3 a} 2^{ab + (b-1) - a}, \\ |\mathcal{D}'_{d,a,b,P}| &\leq ae^{-\mu^3 a} 2^{ab+b-a}. \quad \square \end{aligned}$$

The definition of  $\mathcal{D}_{\delta,p}$ , together with Lemma 6.12, imply that  $\mathcal{D}_{\delta,p} \subset \mathcal{D}'_{\delta,l \sqcap p, l/l \sqcap p, \emptyset}$ ; if we set  $\mu = \mu'(l)$ , Lemma 6.16 implies that

$$|\mathcal{D}_{\delta,p}| \leq l e^{-\mu'(l)^3(l \sqcap p)} 2^{l+(l/l \sqcap p)-l \sqcap p}, \quad (47)$$

which is our second upper bound on  $\mathcal{D}_{\delta,p}$ .

### 6.9. Conclusion

Let us use the two bounds (35) and (47) to estimate the sum described in (33). For  $l \sqcap p \leq l/\ln l$ , we have [by (35)]

$$\begin{aligned} |\mathcal{D}_{\delta,p}| 2^{l \sqcap p} &\leq l e^{-\mu'(l)^3 l} 2^{l+l \sqcap p}, \\ |\mathcal{D}_{\delta,p}| 2^{l \sqcap p} &\leq l e^{-\mu'(l)^3 l} 2^{l+l/\ln l}. \end{aligned} \quad (48)$$

For  $l \sqcap p > l/\ln l$ , we use (47), which implies,

$$\begin{aligned} |\mathcal{D}_{\delta,p}| 2^{l \sqcap p} &\leq l e^{-\mu'(l)^3(l \sqcap p)} 2^{l+l/l \sqcap p}, \\ |\mathcal{D}_{\delta,p}| 2^{l \sqcap p} &\leq l^2 e^{-\mu'(l)^3 l/\ln l} 2^l. \end{aligned} \quad (49)$$

For every term in the sum (33), either (48) or (49) holds. Therefore,

$$\begin{aligned} |E| &\leq \sum_{p=1}^{\lfloor l/2 \rfloor} \max(l e^{-\mu'(l)^3 l} 2^{l+l/\ln l}, l^2 e^{-\mu'(l)^3 l/\ln l} 2^l), \\ |E| &\leq \max(l^2 e^{-\mu'(l)^3 l} 2^{l+l/\ln l}, l^3 e^{-\mu'(l)^3 l/\ln l} 2^l). \end{aligned} \quad (50)$$

From (50), using the definition of  $\mu'$ , we get (after a tedious computation) relation (21).  $\square$

### 6.10. The proof of Capital Lemma 6.3

Let us describe the modifications that the proof of Capital Lemma 6.2 (Sections 6.6–6.9) should undergo in order to become a proof of Capital Lemma 6.3. Note that the function  $\varepsilon$  used in both proofs is the same.

By analogy with the objects  $\delta$  and  $E$  [see (19) and (20)], we define

$$\begin{aligned} \bar{\delta} &= \text{up}(l) + l\varepsilon(l), \\ \bar{E} &= \{S \in \{0, 1\}^l \mid d'(S) > \delta\}. \end{aligned} \quad (51)$$

The property to be proven [corresponding to (21)] can then be expressed by the relation [analogous to (21)]

$$|\bar{E}| \leq \frac{1-\alpha}{2} 2^l. \quad (52)$$

By analogy with (34), we get

$$|\bar{E}| \leq \sum_{p=1}^{\lfloor l/2 \rfloor} \sum_{|D| > \delta} |E_{p,D}|. \quad (53)$$

By analogy with  $\mathcal{D}_{d,p}$  [see (31)], we define for any  $d \in \mathbb{N}$ ,

$$\overline{\mathcal{D}_{d,p}} = \{D \mid |D| > l - d \wedge E_{p,D} \neq \emptyset\}. \quad (54)$$

Then, in the same way as relation (34) is obtained, we get

$$\begin{aligned} |\overline{\mathcal{D}_{\delta,p}}| &\leq \sum_{l-\delta < x \leq l} \binom{l}{x} \\ &\leq \sum_{0 \leq x < \delta} \binom{l}{x}, \end{aligned}$$

which, in turn, leads us to the first upper bound on  $\overline{\mathcal{D}_{\delta,p}}$  [analogous to (35)]:

$$|\overline{\mathcal{D}_{\delta,p}}| \leq l e^{-\mu'(l)^3 l} 2^l. \quad (55)$$

In order to obtain the second upper bound on  $\overline{\mathcal{D}_{\delta,p}}$  [analogous to (47)], we use Lemma 6.12 and get, for all  $i \in [0 \dots l \cap p]$ ,

$$\begin{aligned} D \in \overline{\mathcal{D}_{\delta,p}} &\Rightarrow |((l \cap p)\mathbf{Z} + i) \cap D| \in 2\mathbf{Z}, \\ D \in \overline{\mathcal{D}_{\delta,p}} &\Rightarrow |((l \cap p)\mathbf{Z} + i) \cap ([0 \dots l] - D)| + |((l \cap p)\mathbf{Z} + i) \cap [0 \dots l]| \in 2\mathbf{Z}. \end{aligned} \quad (56)$$

The definition of  $\overline{\mathcal{D}_{\delta,p}}$  [formula (54)] implies that

$$D \in \overline{\mathcal{D}_{\delta,p}} \Rightarrow |[0 \dots l] - D| \leq \delta. \quad (57)$$

From (56) and (57), and from Auxiliary Definition 6.14, we get

$$\overline{\mathcal{D}_{\delta,p}} \subset \{D \subset [0 \dots l] \mid [0 \dots l] - D \in \mathcal{D}'_{\delta, l \cap p, l/l \cap p, [0 \dots l]}\}.$$

Finally, by observing that the function transforming  $D$  (for  $D \subset [0 \dots l]$ ) into  $[0 \dots l] - D$  is bijective, we obtain

$$|\overline{\mathcal{D}_{\delta,p}}| \leq |\mathcal{D}'_{\delta, l \cap p, l/l \cap p, [0 \dots l]}|,$$

and using Lemma 6.16, we get the second upper bound on  $\overline{\mathcal{D}_{\delta,p}}$  (analogous to (47)):

$$|\overline{\mathcal{D}_{\delta,p}}| \leq l e^{-\mu'(l)^3 (l \cap p)} 2^{l + l/l \cap p - l \cap p} \quad (58)$$

The two bounds (55) and (58) enable us to derive (52) in the same way as (21) is obtained in Section 6.9.  $\square$

## References

- [1] P. Bernard, Corpac 900, une autre approche, Tech. Report 06 10 00 MGT 003, TECSI, 29, 1981.
- [2] R.E. Blahut, *Digital Transmission of Information* (Addison-Wesley, 1990).
- [3] S. Kirkpatrick, C.D. Gelatt, Jr. and M.P. Vecchi, Optimization by simulated annealing, *Science* **220** (1983) 671–680.
- [4] R. Lidl and H. Niederreiter, *Introduction to Finite Fields and their Applications* (Cambridge University Press, Cambridge, 1986).
- [5] R.L. Pickholtz, D.L. Schilling, and L.B. Milstein, Theory of spread-spectrum communications—a tutorial, *IEEE Trans Comm.* **30**(5) (1982) 855–884.
- [6] E. Ramis, C. Deschamps, and J. Odoux, *Cours de Mathématiques Spéciales*, Vol. 1 (Masson, Paris, 1979).
- [7] D.V. Sarwate and M.B. Pursley, Crosscorrelation properties of pseudorandom and related sequences, *Proc. IEEE* **68**(5) (1980) 593–619.
- [8] M. Shand, Private communication, February 1991.